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Sharp's Conjecture

the case of local rings with dim nonCM ≤ 1 or dim ≤ 5

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We continue to discuss a conjecture of Sharp on the existence of a dualizing complex from [2] and [3]. For terminologies, definitions and preliminaries, we refer the reader to [2], [3] and [4]. Throughout the note A denotes a d-dimensional local ring with the maximal ideal $\underline{m}$. In this note we show the following two theorems.

Theorem 1. If A has a dualizing complex and $\dim \mathrm{nonCM}(A) \leq 1$, then A is a homomorphic image of a Gorenstein ring.

Theorem 2. If A has a dualizing complex and $\dim A \leq 5$, then A is a homomorphic image of a Gorenstein ring.

In order to show Theorem 1, we make use of Faltings' Macaulayfication ([6]) and the theory of unconditioned strong d-seqences ([7]). If we had Theorem 1, Theorem 2 can be proven by a similar method to that given in [2, §2] and [3, §3].

Now we recall Faltings' Macaulayfication theorem.

Satz 3([6, Satz 3]). Sei B lokaler Ring der Dimension $n+1$, $\underline{n}$ sein maximales Ideal, $I = (x_1,\ldots,x_n) \subset \underline{n}$ ein Ideal mit $\dim B/I = 1$ und $y \in \underline{n}$ mit $\dim B/I+yB = 0$.

Es gelte: i) B ist Quotient eines regulären Ringes.

ii) Für alle minimalen $\underline{p} \in \operatorname{Spek}(B)$ ist $\dim B/\underline{p} = n+1$.

iii) Für alle $\underline{p} \in \operatorname{Spek}(B)$ mit $\underline{p} \not\supseteq I$ ist $B_{\underline{p}}$ Cohen-Macaulay-Ring.

iv) $x_i \in (\operatorname{Ann} H^j_I(B))^{2^n}$ für alle i und alle $j < n$.

v) $x_i \in (\operatorname{Ann} H^j_{\underline{n}}(B))^{2^n}$ für alle i und alle $j \leq n$.

Sei X die Aufblasung des Ideals I in $\operatorname{Spek}(B)$ und $J = I^r\mathcal{O}_X + y\mathcal{O}_X$ ein $\mathcal{O}_X$-Ideal, welches die Faser von X über $\underline{n}$ definiert.

Auf X gelte: vi) $JH^1_{\underline{n}}(\mathcal{O}_X) = 0$.

Sei Y die Aufblasung von X im Ideal J .

Dann ist Y Cohen-Macaulay.

Now we assume that A has a dualizing complex and $\dim \operatorname{nonCM}(A) \leq 1$. We treat the case of $\operatorname{Min}(A) = \operatorname{Assh}(A)$. We note that i) in Satz 3 can be replaced by that B has a dualizing complex ([6, Bemerkungen]). In this case we can take elements $x_1,\ldots,x_{d-1}$, y from $\underline{m}$, for which iii), iv) v) and vi) in Satz 3 hold ([6, Bemerkung a) S.190] and [5]). Furthermore we may assume that r (in Satz 3) is no less than $d-1$ and that $x_1,\ldots,x_{d-1}$ form an unconditioned strong d-sequence in $A_{\underline{p}}$ for every minimal prime ideal $\underline{p}$ of $I = (x_1,\ldots,x_{d-1})$ ([7, 6.19]). Let $L = I^r(I^r + yA)$, $R = \bigoplus_{n\geq 0} L^n \cong A[LT] \subseteq A[T]$ with an indeterminate T and $N = \underline{m}R + R_+$.

Claim: $H^p_N(R)$ is finitely generated for $p \neq d+1$.

It is sufficient to see that R_P is Cohen-Macaulay for every homogeneous prime ideal $P \neq N$. Put $\underline{p} = P \cap A$. First suppose $\underline{p} \neq \underline{m}$. If $\underline{p} \not\supseteq I$, $R_{\underline{p}} \cong A_{\underline{p}}[T]$ is Cohen-Macaulay as so is $A_{\underline{p}}$. If $\underline{p} \supseteq I$, $R_{\underline{p}} \cong \bigoplus_{n \geq 0}(I_{\underline{p}}^r)^n$ is Cohen-Macaulay as $x_1, \ldots, x_{d-1}$ is an unconditioned strong d-sequence in $A_{\underline{p}}$ ([7, 4.1 and 7.10], cf. [3, 1.19]). Now let $\underline{p} = \underline{m}$. As $L^r = (x_1^{2r}, \ldots, x_{d-1}^{2r}, yx_1^r, \ldots, yx_{d-1}^r) \cdot L^{r-1}$ and $P \not\supseteq R_+$, we have $x_i^{2r} \notin P$ for some i or $yx_j^r \notin P$ for some j . Let $P \not\ni x_1^{2r}$. We put $t = x_1^{2r}T$, $S = R[1/t]$, $B = S_0$ and $Q = PS \cap B$ $(\supseteq \underline{m}B)$. Since $S = B[t, 1/t]$ and t is algebraically independent over B , S_{PS} is Cohen-Macaulay if and only if so is B_Q . Hence it is sufficient to show that B_M is Cohen-Macaulay for every maximal ideal M of B containing $\underline{m}B$. $B = S_0 = A[x/x_1^{2r} \mid x \in L] = A[x_2/x_1, \ldots, x_{d-1}/x_1, y/x_1^r]$. Satz 3 asserts that B is Cohen-Macaulay. In the case of $P \not\ni yx_1^r$, the proof is similar to the above.

Hence we have Theorem 1 (cf. [2, Proof of 3.10] and [3, Proof of 4.11]).

We mention that the same theorem as Satz 3 (hence as Theorem 1) holds for a semi-local ring $(B, \underline{n}_1, \ldots, \underline{n}_t)$ if all $\underline{n}_1, \ldots, \underline{n}_t$ appear in the same degree term of a fundamental dualizing complex and every maximal chain of prime ideals has the same length.

Corollary to Theorem 1. If A has a dualizing complex and A is (S_{d-2}), then A is a homomorphic image of a Gorenstein ring.

Now we prove Theorem 2. Let $d = 5$. (See [2, §2] or [3, §3] for the case of $d \leq 4$.) Suppose that the assertion is false. Then, by [2, 2.1] or [3, 3.1], there is a 5-dimensional local ring A such that A has a dualizing complex, is not a homomorphic image of a Gorenstein ring and is (S_2). A is not (S_3) by Corollary above. Then $T(A) := \{ \underline{p} \in \mathrm{Spec}(A) \mid \mathrm{depth}\, A_{\underline{p}} = 2 < \dim A_{\underline{p}} \}$ is not empty. Let I be an ideal such that $V(I) = \mathrm{nonCM}(A)$. As A is (S_2), height $I \geq 3$. There is an A-regular sequence a, b in I. We have $T(A) \subset \mathrm{Ass}(A/(a,b))$. We put $s(A) = \max \{ \dim A_{\underline{p}} \mid \underline{p} \in T(A) \}$, $T_0(A) = \{ \underline{p} \in T(A) \mid \dim A_{\underline{p}} = s(A) \}$ and $T_1(A) = T(A) \setminus T_0(A)$. Consider all such local rings, and take a local ring A from them whose $s(A)$ is the smallest. As A is (S_2), $H^2_{\underline{p}A_{\underline{p}}}(A_{\underline{p}})$ is of finite length for every $\underline{p}$ in $\mathrm{Spec}(A)$ with $\dim A_{\underline{p}} \geq 3$. Hence there is a non zero divisor $x \in \bigcap\{\underline{p} \mid \underline{p} \in T_0(A)\} \setminus \bigcup\{\underline{p} \mid \underline{p} \in T_1(A)\}$ such that $xH^2_{\underline{p}A_{\underline{p}}}(A_{\underline{p}}) = 0$ for every $\underline{p}$ in $T_0(A)$. Let $C = \mathrm{Hom}_{A/xA}(K_{A/xA}, K_{A/xA})$. By the fact we mentioned before Corollary to Theorem 1, there exists a Gorenstein semi-local ring G such that $\mathrm{Max}(G) = \{ \underline{n} \cap G \mid \underline{n} \in \mathrm{Max}(C) \}$, every maximal chain of prime ideals in G has length 5, the length of a fundamental dualizing comlex of G is equal to 5 and C is a homomorphic image of G. Let B be the fibre product of $A \to C$ and $G \to C$. We have an exact sequence of B-modules $0 \longrightarrow B \longrightarrow A \oplus G \longrightarrow C \longrightarrow 0$. By the same argument as in Proof of [2, 2.3] or [3, 3.2], it is known that B is a 5-dimensional local ring with the maximal ideal $\underline{m} \cap B$ and B has a dualizing complex. As A is a homomorphic image of B, B is not a homomorphic image of a Gorenstein ring and not (S_3). B is (S_2). Hence $T(B) \neq \emptyset$, and $s(B) \geq s(A)$ by the choice of A. Take P from $T_0(B)$. We have depth $B_P = 2$. If $C_P = 0$, $B_P \cong A_P$ as G is Gorenstein. Hence $PA \in T_0(A)$, a contradiction as $PA \not\ni x$. Therefore $C_P \neq 0$. Put $\dim C_P = t$. Then $\dim B_P = \dim A_P = \dim G_P = t + 1 = s(B) \geq s(A) > 2$. From the exact sequence $0 \longrightarrow B_P \longrightarrow A_P \oplus G_P \longrightarrow C_P \longrightarrow 0$, we have depth $A_P = 2$ as depth $B_P = 2$, depth $G_P = t+1 > 2$ and depth $C_P \geq 2$. Therefore $PA \in T_0(A)$ and $s(B) = s(A)$. Hence $xH^2_{PA_P}(A_P) = 0$ and $H^2_{PA_P}(A_P) \to H^2_{PA_P}(A_P/xA_P)$ is injective. It is known that A_P/xA_P is (S_2) at every non-maximal prime ideal. Hence $H^2_{PA_P}(A_P/xA_P) \to H^2_{PA_P}(C_P)$ is injective (cf. [1, Proposition 2]). From the exact sequence $0 = H^1_{PB_P}(C_P) \longrightarrow H^2_{PB_P}(B_P) \longrightarrow H^2_{PB_P}(A_P \oplus G_P) \cong H^2_{PA_P}(A_P) \longrightarrow H^2_{PB_P}(C_P)$, we have $H^2_{PB_P}(B_P) = 0$, which contradicts depth $B_P = 2$. Now the proof is completed.

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